

Differentials on the arc space

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ABSTRACT. The paper provides a description of the sheaves of Kähler differentials of the arc space and jet schemes of an arbitrary scheme where these sheaves are computed directly from the sheaf of differentials of the given scheme. Several applications on the structure of arc spaces are presented.

1. Introduction

The work of Greenberg, Nash, Kolchin, and Denef–Loeser [Gre66, Nas95, Kol73, DL99] has set the basis for our understanding of the structure of arc spaces and their connections to singularities and birational geometry. Most of the focus in these studies is on the reduced structure of arc spaces and their underlying topological spaces, and little is known about their scheme structure. Notable studies in this direction are those of Reguera [Reg06, Reg09], recently continued in [Reg, MR].

This paper can be viewed as a continuation of these studies. In the first part we describe the sheaves of Kähler differentials of the arc space and of the jet schemes, which answers a question raised by Ein. The second part of the paper is devoted to applications of the resulting formulas. The approach leads to new results as well as simpler and more direct proofs of some of the theorems in the literature, and provides a new way of understanding some of the fundamental properties of the theory.

In the first part of the paper we work over an arbitrary base scheme S . Let X be a scheme over S . The *arc space* of X over S , denoted $(X/S)_\infty$, is the universal object in the category of schemes parametrizing arcs on X over S . As any moduli space, it comes equipped with a tautological family, known as the *universal arc*:

$$\begin{array}{ccc} U_\infty & \xrightarrow{\gamma_\infty} & X \\ \rho_\infty \downarrow & & \downarrow \\ (X/S)_\infty & \longrightarrow & S \end{array}$$

The universal arc is given by $U_\infty = (X/S)_\infty \hat{\times} \mathrm{Spf} \mathbb{Z}[[t]]$. On U_∞ , we construct a sheaf \mathcal{P}_∞ whose $\mathcal{O}_{(X/S)_\infty}$ -dual is \mathcal{O}_{U_∞} . The sheaf \mathcal{P}_∞ plays the role of “kernel” in the following formula which relates the sheaf of differentials of $(X/S)_\infty$ directly to the sheaf of differentials of X .

Theorem A (Kähler differentials on arc spaces). *There is a natural isomorphism*

$$\Omega_{(X/S)_\infty/S} \simeq \rho_{\infty*}(\gamma_\infty^*(\Omega_{X/S}) \otimes \mathcal{P}_\infty).$$

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We have a similar result for jet schemes. If $(X/S)_n$ denotes the n -th *jet scheme* of X over S , then we have a *universal jet* $(X/S)_n \xleftarrow{\rho_n} U_n \xrightarrow{\gamma_n} X$, and we construct a sheaf \mathcal{P}_n on U_n whose $\mathcal{O}_{(X/S)_n}$ -dual is \mathcal{O}_{U_n} . In this case, it turns out that \mathcal{P}_n is the $\mathcal{O}_{(X/S)_n}$ -dual of \mathcal{O}_{U_n} and is trivial as an \mathcal{O}_{U_n} -module, and the description of the sheaf of differentials gets simplified.

Theorem B (Kähler differentials on jet schemes). *There are natural isomorphisms*

$$\Omega_{(X/S)_n/S} \simeq \rho_{n*}(\gamma_n^*(\Omega_{X/S}) \otimes \mathcal{P}_n) \simeq \rho_{n*}(\gamma_n^*(\Omega_{X/S})).$$

We apply these theorems to study the structure of arc spaces.

The starting point is the description of the fibers of the sheaf of differentials of the jet schemes $(X/S)_n$ at liftable jets. Suppose that X is a scheme of finite type over S , and let $\alpha \in (X/S)_\infty$ be an arc. For $n \gg 1$, let $\alpha_n \in (X/S)_n$ be the truncation of α and L_n the residue field of α_n . Using the above theorems, we determine an isomorphism

$$\Omega_{(X/S)_n/S} \otimes L_n \simeq (L_n[t]/(t^{n+1}))^d \oplus \bigoplus_{i \geq d} L_n[t]/(t^{e_i}).$$

Here the number d and the sequence $\{e_i\}$ are certain Fitting-theoretic invariants of the pull-back of $\Omega_{X/S}$ by α (see Section 6 for the precise definition of these invariants and Theorem 7.2 for the precise statement). If X is a reduced and equidimensional scheme of finite type over a field k , and α is not fully contained in the singular locus of X , then we have $d = \dim X$ and $\sum_{i \geq d} e_i = \text{ord}_\alpha(\text{Jac}_X)$ where Jac_X is the Jacobian ideal of X , and the above isomorphism recovers in this case one of the main results of [dFD14].

These results have several applications. For simplicity, for the remainder of the introduction we shall assume that X is a variety defined over a perfect field k , though more general results are obtained in the paper. In this setting, the arc space and the jet schemes are denoted by X_∞ and X_n .

A natural way of studying the structure of the arc space of X is to analyze its (not necessarily closed) points. Given a point $\alpha \in X_\infty$, we are interested in two invariants: the *embedding dimension*

$$\text{emb. dim}(\mathcal{O}_{X_\infty, \alpha})$$

of the local ring at α , and the *jet codimension* of α in X_∞ , which is defined by

$$\text{jet. codim}(\alpha, X_\infty) := \lim_{n \rightarrow \infty} ((n+1) \dim X - \dim \overline{\{\alpha_n\}}).$$

These and related invariants have been studied in the literature (e.g., see [ELM04, dFEI08, dFM15, Reg, MR]). Both numbers provide measures of the “size” of the point. Note however that while the embedding dimension is computed on the arc space (with its scheme structure), the jet codimension is computed from the truncations of the arc and only depends on the reduced structure of the jet schemes.

Our first application is that on varieties (and more generally, on reduced schemes of finite type) these two invariants measure the same quantity.

Theorem C (Embedding dimension as jet codimension). *Given a variety X over a perfect field, we have*

$$\text{emb. dim}(\mathcal{O}_{X_\infty, \alpha}) = \text{jet. codim}(\alpha, X_\infty)$$

for every $\alpha \in X_\infty$.

One of the most important results on arc spaces is the Birational Transformation Rule [Kon95, DL99]. It implies the change-of-variable formula in motivic integration

[Kon95, DL99, Bat99, Loo02] and has been applied to study invariants of singularities in birational geometry (e.g., see [Mus01, Mus02, EMY03, EM04, ISW12, Ish13, IR13, dFD14, EI15, Zhu15]). Using our description of the sheaf of differentials, we obtain the following variant.

Theorem D (Birational transformation rule). *Given a proper birational map $f: Y \rightarrow X$ between two varieties over a perfect field, we have*

$$\text{emb. dim}(\mathcal{O}_{Y_\infty, \beta}) \leq \text{emb. dim}(\mathcal{O}_{X_\infty, f_\infty(\beta)}) \leq \text{emb. dim}(\mathcal{O}_{Y_\infty, \beta}) + \text{ord}_\beta(\text{Jac}_f)$$

for every $\beta \in Y_\infty$, where Jac_f is the Jacobian of f . Moreover, if Y is smooth at $\beta(0)$, then

$$\text{emb. dim}(\mathcal{O}_{X_\infty, f_\infty(\beta)}) = \text{emb. dim}(\mathcal{O}_{Y_\infty, \beta}) + \text{ord}_\beta(\text{Jac}_f).$$

The connection between this theorem and Denef–Loeser’s Birational Transformation Rule becomes evident once one rewrites the second formula of Theorem D using Theorem C, which gives the formula

$$\text{jet. codim}(f_\infty(\beta), X_\infty) = \text{jet. codim}(\beta, Y_\infty) + \text{ord}_\beta(\text{Jac}_f)$$

for any resolution of singularities $f: Y \rightarrow X$. Although it does not retain all the information provided in [DL99] that is necessary for the change-of-variable formula in motivic integration, this formula suffices for all known applications to the study of singularities in birational geometry.

Our next application regards the *stable points* of the arc space. By definition, these are the generic points of the irreducible constructible subsets of X_∞ (see Section 11 for a discussion of the notion of constructibility in arc spaces). Stable points and their local rings have been extensively studied in [Reg06, Reg09, Reg, MR]. The following theorem can be viewed as providing a characterization of stable points that are not fully contained, as arcs, within the singular locus of X .

Theorem E (Characterization of finite embedding dimension). *Given a variety X (or, more generally, any scheme of finite type) over a perfect field, and a point $\alpha \in X_\infty$, we have*

$$\text{emb. dim}(\mathcal{O}_{X_\infty, \alpha}) < \infty$$

if and only if α is a stable point and is not contained in $(\text{Sing } X)_\infty$.

It follows from general properties of local rings that the embedding dimension at a point $\alpha \in X_\infty$ is finite if and only if the completion of the local ring is Noetherian. The fact that the completion of the local ring at a stable point $\alpha \in X_\infty \setminus (\text{Sing } X)_\infty$ is Noetherian is a theorem of Reguera [Reg06, Reg09], and we get a new proof of this important result. It is the key ingredient in the proof of the Curve Selection Lemma, which plays an essential role in the recent progress on the Nash problem (e.g., see [FdBPP12, dFD16]).

There are examples in positive characteristics of varieties X whose arc space X_∞ has irreducible components that are fully contained in $(\text{Sing } X)_\infty$, and Theorem E implies that X_∞ has infinite embedding codimension at the generic points of such components (see Remark 11.6). One should contrast this with the main theorem of [GK00, Dri], which can be interpreted as saying that the completion of the local ring of X_∞ at any k -valued point that is not contained in $(\text{Sing } X)_\infty$ has finite embedding codimension.

A special class of stable points is given by what we call the *maximal divisorial arcs*. By definition, these are the arcs $\alpha \in X_\infty$ whose associated valuation ord_α is a divisorial valuation and that are maximal (with respect to specialization) among all arcs defining

the same divisorial valuation. Equivalently, they are the generic points of the maximal divisorial sets defined in [ELM04, dFEI08, Ish08]. Our final application gives the following result.

Theorem F (Embedding dimension at maximal divisorial arcs). *Let X be a variety over a perfect field, $f: Y \rightarrow X$ a proper birational morphism from a normal variety Y , E a prime divisor on Y , and q a positive integer. If $\alpha \in X_\infty$ is the maximal divisorial arc corresponding to the divisorial valuation $q \operatorname{ord}_E$, then*

$$\operatorname{emb. dim}(\mathcal{O}_{X_\infty, \alpha}) = q(\operatorname{ord}_E(\operatorname{Jac}_f) + 1).$$

The quantity $\operatorname{ord}_E(\operatorname{Jac}_f)$ is also known as the *Mather discrepancy* of E over X , and denoted by $\widehat{k}_E(X)$ [dFEI08]. In view of Theorem C, Theorem F recovers [dFEI08, Theorem 3.8]. The theorem is also closely related to a recent result of Mourtada and Reguera [Reg, MR] which states that with the same assumptions as in Theorem F, if the field has characteristic 0 then

$$\operatorname{emb. dim}(\widehat{\mathcal{O}_{X_\infty, \alpha}}) = \operatorname{emb. dim}(\mathcal{O}_{(X_\infty)_{\operatorname{red}}, \alpha}) = q(\operatorname{ord}_E(\operatorname{Jac}_f) + 1).$$

As a by-product, we deduce that, under these assumptions, the local ring $\mathcal{O}_{X_\infty, \alpha}$ has the same embedding dimension as its completion $\widehat{\mathcal{O}_{X_\infty, \alpha}}$ (a non-obvious fact since $\mathcal{O}_{X_\infty, \alpha}$ is not necessarily Noetherian) as well as its reduction $\mathcal{O}_{(X_\infty)_{\operatorname{red}}, \alpha}$.

Proofs of the results stated in the introduction are located as follows: both statements in Theorems A and B are contained in Theorem 5.3, Theorem C follows from Corollary 8.10, Theorem D combines the statements of Theorems 9.2 and 9.3, Theorem E is proved in Theorem 11.5, and Theorem F follows from Theorem 10.4.

All proofs in this paper rely only on the definition of arc space and basic facts in commutative algebra, with the only exception occurring in the last section where, in order to study stable points, we apply a lemma describing the structure of the truncation maps between the sets of liftable jets due to [DL99, EM09].

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2. Conventions

Throughout the paper, all rings are assumed to be commutative with identity. Unless otherwise specified, rings are regarded with the discrete topology; however, power series rings of the form $R[[t]]$ are considered as complete topological rings. For topological modules M and N over a topological ring R , we define their completed tensor product, denoted by $M \hat{\otimes}_R N$, as the completion of the ordinary tensor product $M \otimes_R N$. We will mostly encounter completed tensor products of the form $M \hat{\otimes}_R A[[t]]$, where R is a ring, A is an R -algebra, and M is an R -module, all with the discrete topology.

We fix a base scheme S and work on the category of schemes over S . Given an object X in this category, we do not impose any condition on the morphism $X \rightarrow S$. However, starting with Section 8 we will assume that $S = \operatorname{Spec} k$ where k is a perfect field, and mostly focus on schemes of finite type over k .

We also need to consider formal schemes over S . For our purposes, it will be enough to consider the notion of formal scheme introduced in [EGA I]. In fact, we will only consider formal schemes of the form $X \hat{\times}_{\mathbb{Z}} \mathrm{Spf} \mathbb{Z}[[t]]$ where X is an ordinary scheme. Here and in the sequel we use the symbol $\hat{\times}$ to denote the product in the category of formal schemes; this emphasizes the fact that it corresponds to the completed tensor product $\hat{\otimes}$ at the level of topological rings. In particular,

$$\mathrm{Spec} R \hat{\times}_{\mathbb{Z}} \mathrm{Spf} \mathbb{Z}[[t]] = \mathrm{Spf}(R \hat{\otimes}_{\mathbb{Z}} \mathbb{Z}[[t]]) = \mathrm{Spf} R[[t]].$$

Unless otherwise stated, the letters m and n will be used to denote elements in the set $\{0, 1, 2, \dots, \infty\}$.

3. Generalities on arcs and jets

Let X be an arbitrary scheme over a base scheme S . The jet schemes and the arc space of X over S are defined, in this generality, in [Voj07], to which we refer for more details and proofs; see also [IK03, EM09].

For every non-negative integer n , the n -jet scheme $(X/S)_n$ of X over S represents the functor from S -schemes to sets given by

$$Z \mapsto \mathrm{Hom}_S(Z \times_{\mathbb{Z}} \mathrm{Spec} \mathbb{Z}[t]/(t^{n+1}), X),$$

while the arc space $(X/S)_{\infty}$ of X over S represents the functor from S -schemes to sets given by

$$Z \mapsto \mathrm{Hom}_S(Z \hat{\times}_{\mathbb{Z}} \mathrm{Spf} \mathbb{Z}[[t]], X).$$

A point of $(X/S)_n$ is called an n -jet of X , and a point of $(X/S)_{\infty}$ an arc of X . Note that an arc $\alpha \in (X/S)_{\infty}$ can be equivalently thought as a map $\mathrm{Spf} L[[t]] \rightarrow X$ or a map $\mathrm{Spec} L[[t]] \rightarrow X$ (both defined over S), where L is the residue field of α , see Lemma 3.1 below. The advantage of considering α as a map $\mathrm{Spec} L[[t]] \rightarrow X$ is that it allows us to talk about the *generic point of the arc*, by which we mean the image $\alpha(\eta) \in X$ of the generic point η of $\mathrm{Spec} L[[t]]$. We will denote by $\alpha(0)$ the image of the closed point of $\mathrm{Spec} L[[t]]$.

If $S = \mathrm{Spec} R$ where R is a ring, then one can replace \mathbb{Z} with R in the above formulas. If $S = \mathrm{Spec} k$ where k is a field, then we will simply denote $(X/S)_n$ by X_n and $(X/S)_{\infty}$ by X_{∞} .

For any n in $\{0, 1, 2, \dots, \infty\}$, the scheme $(X/S)_n$ is equipped with a universal family

$$\begin{array}{ccc} U_n & \xrightarrow{\gamma_n} & X \\ \rho_n \downarrow & & \downarrow \\ (X/S)_n & \longrightarrow & S \end{array} \quad (3a)$$

For n finite, the family is given by

$$U_n = (X/S)_n \times_{\mathbb{Z}} \mathrm{Spec} \mathbb{Z}[t]/(t^{n+1}),$$

and is called the *universal n -jet*. For $n = \infty$, it is given by

$$U_{\infty} = (X/S)_{\infty} \hat{\times}_{\mathbb{Z}} \mathrm{Spf} \mathbb{Z}[[t]]$$

and is called the *universal arc*. Notice that U_{∞} is a formal scheme.

We will use the following notations for the natural truncation maps:

$$\pi_n: (X/S)_{\infty} \rightarrow (X/S)_n, \quad \pi_{m,n}: (X/S)_m \rightarrow (X/S)_n, \quad \psi_n: (X/S)_n \rightarrow X.$$

Note that γ_n is different from the composition map $\psi_n \circ \rho_n$. Furthermore, observe that there is no natural map between U_m and U_n when $m > n$. The natural map is

$$\mu_{m,n}: U_n \times_{(X/S)_n} (X/S)_m \longrightarrow U_m,$$

where the fiber product is taken with respect to the maps ρ_n and $\pi_{m,n}$.

It will be useful to have notation in place for the affine case. The following basic properties are proved in [Voj07, Corollary 1.8 and Theorem 4.5].

Lemma 3.1. *Assume that $S = \operatorname{Spec} R$ for a ring R and $X = \operatorname{Spec} A$ for an R -algebra A . Then $(X/S)_n$ is affine for all n . If we write $(X/S)_n = \operatorname{Spec} A_n$, then for every R -algebra C we have*

$$(X/S)_n(C) = \operatorname{Hom}_{R\text{-alg}}(A_n, C) = \operatorname{Hom}_{R\text{-alg}}(A, C[t]/(t^{n+1})),$$

if n is finite, and

$$(X/S)_\infty(C) = \operatorname{Hom}_{R\text{-alg}}(A_\infty, C) = \operatorname{Hom}_{R\text{-alg}}(A, C[[t]]),$$

Moreover, A_n is characterized by the above property.

More explicitly, jet schemes and arc spaces can be defined using Hasse–Schmidt derivations: denoting by $A_n := \operatorname{HS}_{A/R}^n$ the A -algebra given in [Voj07, Definition 1.3] and defining $B_n := A_n[t]/(t^{n+1})$ when n is finite and $B_\infty := A_\infty[[t]]$ when $n = \infty$, we have

$$(X/S)_n = \operatorname{Spec} A_n \quad \text{and} \quad U_n = \operatorname{Spf} B_n.$$

The universal jet (or arc) is given by

$$\begin{array}{ccc} B_n & \xleftarrow{\gamma_n^\#} & A \\ \rho_n^\# \uparrow & & \uparrow \\ A_n & \longleftarrow & R \end{array} \quad (3b)$$

where the map $\rho_n^\#$ is the natural inclusion $A_n \subset B_n$, and the map $\gamma_n^\#$ is defined by

$$\gamma_n^\#(f) = \sum_{p=0}^n D_p(f) t^p$$

where $(D_p)_{p=0}^n$ is the universal Hasse–Schmidt derivation.

We will consider in B_n the A -module structure given by $\gamma_n^\#$ and the A_n -module structure given by $\rho_n^\#$. Notice that B_n has a second A -module structure (induced from the inclusion $A \subset A_n \subset B_n$), but we will have no use for it.

For $m > n$, the map $\mu_{m,n}: U_n \times_{(X/S)_n} (X/S)_m \rightarrow U_m$ is defined by the natural projection

$$\mu_{m,n}^\#: B_m = A_m[t]/(t^{m+1}) \longrightarrow B_n \otimes_{A_n} A_m = A_m[t]/(t^{n+1})$$

when m is finite, and

$$\mu_{\infty,n}^\#: B_\infty = A_\infty[[t]] \longrightarrow B_n \otimes_{A_n} A_\infty = A_\infty[t]/(t^{n+1})$$

when $m = \infty$.

Remark 3.2. If X is a quasi-compact and quasi-separated scheme over a field k , then it follows from the results in [Bha16] that the functor of points of the arc space X_∞ can also be described as

$$X_\infty(C) = \operatorname{Hom}_k(\operatorname{Spec} C[[t]], X)$$

for any k -algebra C . Notice that this description avoids the use of formal schemes, and in particular it gives a universal arc which is defined as an ordinary scheme. The category of quasi-compact and quasi-separated schemes is probably large enough to contain all arc spaces of geometric interest, but the theory developed in [Bha16] is very delicate, and we preferred to avoid relying on it. Our results are local in nature, so it is enough for us to have an analogue of the above formula in the affine case. This is precisely the content of Lemma 3.1, which is an elementary fact from commutative algebra. Our results do not require the quasi-compact and quasi-separated conditions.

4. The sheaves \mathcal{P}_n

The goal of this section is to define the sheaves \mathcal{P}_n appearing in Theorems A and B. We start by looking at the affine case. We continue with the notation introduced in Section 3, so that given an R -algebra A we have $A_n = \mathrm{HS}_{A/R}^n$, $B_n = A_n[t]/(t^{n+1})$ when n is finite, and $B_\infty = A_\infty[[t]]$.

Definition 4.1. For any n , we define P_n to be the B_n -module given by

$$P_n := t^{-n} A_n[t]/t A_n[t]$$

when n is finite and

$$P_\infty := A_\infty((t))/t A_\infty[[t]]$$

when $n = \infty$.

As A_n -modules, we have $B_n = \prod_{i=0}^n A_n t^i$ and $P_n = \bigoplus_{j=0}^n A_n t^{-j}$. It is convenient to view an element $b \in B_n$ as a power series $b = \sum_{i=0}^n a_i t^i$ (a polynomial if n is finite), and an element $p \in P_n$ as a polynomial $p = \sum_{j=0}^n a'_j t^{-j}$. With this in mind, we can view the B_n -module structure as follows: the action of an element $b \in B_n$ on an element $p \in P_n$ is simply given by the product $b \cdot p$ of the two series, modulo $t A_n[[t]]$.

Note that the A_n -module $\mathrm{Hom}_{A_n}(P_n, A_n)$ has a natural B_n -module structure given by precomposition. That is, given $b \in B_n$ and $\phi: P_n \rightarrow A_n$, we define $b \cdot \phi$ to be the homomorphism $P_n \rightarrow A_n$ defined by $(b \cdot \phi)(p) := \phi(b \cdot p)$.

Lemma 4.2. *For every n , there is a canonical isomorphism $B_n \simeq \mathrm{Hom}_{A_n}(P_n, A_n)$ as B_n -modules.*

Proof. Since $B_n = \prod_{i=0}^n A_n t^i$ and $P_n = \bigoplus_{j=0}^n A_n t^{-j}$, there is a canonical isomorphism of A_n -modules $B_n \simeq \mathrm{Hom}_{A_n}(P_n, A_n)$ given by

$$b = \sum_{i=0}^n a_i t^i \mapsto \left(\phi_b: p = \sum_{j=0}^n a'_j t^{-j} \mapsto \sum_{i=0}^n a_i a'_{-i} \right),$$

and it is immediate to check that this isomorphism is compatible with the respective B_n -module structures. \square

Remark 4.3. Lemma 4.2 generalizes to all A_n -modules, in the following way. For every A_n -module M , the space $\mathrm{Hom}_{A_n}(P_n, M)$ has a natural B_n -module structure given by precomposition, and there is a canonical isomorphism

$$M \hat{\otimes}_{A_n} B_n \simeq \mathrm{Hom}_{A_n}(P_n, M)$$

as B_n -modules. The proof follows the same arguments of the proof of Lemma 4.2, once one observes that $M \hat{\otimes}_{A_n} B_n = \prod_{i=0}^n M t^i$.

Remark 4.4. When n is finite, we can view $\{t^{-j}\}_{j=0}^n$ as the dual basis of $\{t^i\}_{i=0}^n$, and we have $P_n \simeq \text{Hom}_{A_n}(B_n, A_n)$. Note, though, that P_∞ is not the A_∞ -dual of B_∞ .

Lemma 4.5. *For finite n , the morphism that sends t^{-j} to t^{-j+n} gives an isomorphism of B_n -modules between P_n and B_n . By contrast, P_∞ and B_∞ are not isomorphic, not even as A_∞ -modules.*

Proof. Multiplication by t^n clearly gives an isomorphism of A_n -modules $P_n \simeq B_n$, and one can check that this is compatible with the B_n -module structures. The last assertion is also clear since $P_\infty \simeq A_\infty[t]$ (as A_∞ -module) whereas $B_\infty = A_\infty[[t]]$. \square

Remark 4.6. For $m > n$, the homomorphism $\mu_{m,n}^\sharp: B_m \rightarrow B_n \otimes_{A_n} A_m$ defining the morphism $\mu_{m,n}: U_n \times_{(X/S)_n} (X/S)_m \rightarrow U_m$ corresponds, via the duality given in Lemma 4.2, to the inclusion

$$P_n \otimes_{A_n} A_m \rightarrow P_m$$

that sends t^{-j} in $P_n \otimes_{A_n} A_m$ to t^{-j} in P_m . When m is finite, this inclusion corresponds via the natural isomorphisms $P_n \simeq B_n$ and $P_m \simeq B_m$ to the homomorphism $B_n \otimes_{A_n} A_m \rightarrow B_m$ given by multiplication by t^{m-n} .

The definition of P_n globalizes as follows. Given an arbitrary morphism of schemes $X \rightarrow S$, source and target can be covered by affine charts $\text{Spec } A \subset X$ and $\text{Spec } R \subset S$ so that the morphism is determined by gluing affine morphisms $\text{Spec } A \rightarrow \text{Spec } R$. For every n , let U_n be the universal family given in Eq. (3a). Then the sheaves P_n constructed above for the corresponding charts $\text{Spec } B_n \subset U_n$ glue together to give a sheaf \mathcal{P}_n on U_n .

For every n , there is a natural isomorphism

$$\rho_{n*}(\mathcal{O}_{U_n}) \simeq \text{Hom}_{\mathcal{O}_{(X/S)_n}}(\rho_{n*}(\mathcal{P}_n), \mathcal{O}_{(X/S)_n}).$$

Moreover, the right hand side has a natural \mathcal{O}_{U_n} -module structure given by precomposing with the \mathcal{O}_{U_n} -module action on \mathcal{P}_n , and with this structure is isomorphic to \mathcal{O}_{U_n} . Furthermore, if n is finite then we have $\mathcal{P}_n \simeq \mathcal{O}_{U_n}$. All these statements can be checked locally on X , and therefore it suffices to consider the case where $X = \text{Spec } A$ and $S = \text{Spec } R$, where they reduce to Lemmas 4.2 and 4.5.

The analysis done in the affine case can be carried out in an identical way in this more general setting. In particular, for each $m > n$ we get a natural injective morphism

$$\pi_{m,n}^*(\rho_{n*}(\mathcal{P}_n)) \rightarrow \rho_{m*}(\mathcal{P}_m).$$

If m and n are finite, then $\mathcal{P}_m \simeq \mathcal{O}_{U_m}$ and $\mathcal{P}_n \simeq \mathcal{O}_{U_n}$, and the above injection is conjugate to the morphism $\pi_{m,n}^*(\rho_{n*}(\mathcal{O}_{U_n})) \rightarrow \rho_{m*}(\mathcal{O}_{U_m})$ given by multiplication by t^{m-n} .

5. Derivations and differentials

In this section we prove the description of $\Omega_{(X/S)_n}$ stated in Theorems A and B. We continue with the notations introduced in the previous section, so that given an R -algebra A we have $A_n = \text{HS}_{A/R}^n$, $B_n = A_n[t]/(t^{n+1})$ when n is finite, and $B_\infty = A_\infty[[t]]$. As before, we regard B_n as an A_n -module via ρ_n^\sharp and as an A -module via γ_n^\sharp , where these maps are defined in Eq. (3b).

Lemma 5.1. *Let R be a ring and A an R -algebra. Let M be an A_n -module, and consider $M \hat{\otimes}_{A_n} B_n$ with the A -module structure induced from the A -module structure on B_n (notice that $\hat{\otimes}_{A_n} = \otimes_{A_n}$ when n is finite). Then there is a natural isomorphism*

$$\text{Der}_R(A_n, M) \simeq \text{Der}_R(A, M \hat{\otimes}_{A_n} B_n).$$

If $m > n$ and M is an A_m -module, then the natural map $\mathrm{Der}_R(A_m, M) \rightarrow \mathrm{Der}_R(A_n, M)$ corresponds via the above isomorphism to the map induced by $\mu_{m,n}^\sharp: B_m \rightarrow B_n \otimes_{A_n} A_m$.

Proof. To treat the cases of arcs and jets at the same time, we will identify $R[[t]]$ with $R[t]/(t^{n+1})$ when $n = \infty$.

Fix an A_n -module M as in the statement of the lemma, and consider the A_n -module $A_n \oplus \varepsilon M$ with the A_n -algebra structure defined by $(r \oplus \varepsilon m) \cdot (r' \oplus \varepsilon m') = (rr' \oplus \varepsilon(rm' + r'm))$. The symbol ε should be thought as a variable with $\varepsilon^2 = 0$. Since $A_n \hat{\otimes}_R R[t]/(t^{n+1}) = B_n$, Lemma 3.1 gives a natural isomorphism

$$\mathrm{Hom}_{R\text{-alg}}(A_n, A_n \oplus \varepsilon M) \simeq \mathrm{Hom}_{R\text{-alg}}(A, B_n \oplus \varepsilon(M \hat{\otimes}_{A_n} B_n)).$$

The two modules of differentials that we are interested in are mapped into each other via this isomorphism. More precisely, we have

$$\begin{aligned} \mathrm{Der}_R(A_n, M) &\simeq \{ \phi \in \mathrm{Hom}_{R\text{-alg}}(A_n, A_n \oplus \varepsilon M) \mid \phi = \mathrm{id}_{A_n} \bmod \varepsilon \} \\ &\simeq \{ \phi \in \mathrm{Hom}_{R\text{-alg}}(A, B_n \oplus \varepsilon(M \hat{\otimes}_{A_n} B_n)) \mid \phi = \gamma_n^\sharp \bmod \varepsilon \} \\ &\simeq \mathrm{Der}_R(A, M \hat{\otimes}_{A_n} B_n). \end{aligned}$$

For the second statement of the lemma, it suffices to note that the map $\mathrm{Der}_R(A_m, M) \rightarrow \mathrm{Der}_R(A_n, M)$ corresponds, via the above isomorphisms, to the map

$$\mathrm{Hom}_{R\text{-alg}}(A, B_m \oplus \varepsilon(M \hat{\otimes}_{A_m} B_m)) \rightarrow \mathrm{Hom}_{R\text{-alg}}(A, B_n \oplus \varepsilon(M \hat{\otimes}_{A_n} B_n))$$

induced by the projection $R[t]/(t^{m+1}) \rightarrow R[t]/(t^{n+1})$, and the latter is exactly the projection that induces $\mu_{m,n}^\sharp$.

All the isomorphisms in the proof are functorial with respect to all the data involved, and therefore the resulting isomorphisms are natural. \square

Remark 5.2. The previous lemma is the algebraic incarnation of an intuitive geometric fact about tangent vectors on arc spaces and jet schemes. For concreteness, we look at the case of arcs when $R = k$ is a field. Consider a point α in X_∞ with residue field L , here regarded as an A_∞ -module. Then an element of $\mathrm{Der}_k(A_\infty, L)$ corresponds to a tangent vector to X_∞ at α . Using the isomorphism of Lemma 5.1, this tangent vector gets identified with an element of $\mathrm{Der}_k(A, L[[t]])$, which corresponds to a vector field on X along the image of the arc α . These types of identifications are expected for moduli spaces of maps. For example, given two smooth projective varieties X and Y , we can consider the space $\mathcal{M} = \mathrm{Mor}(Y, X)$ parametrizing morphisms from Y to X . Then, for a morphism $f: Y \rightarrow X$, we have the well-known formula

$$T_{\mathcal{M},f} \simeq H^0(Y, f^*T_X),$$

which is analogous to Lemma 5.1.

Theorem 5.3. *Let $X \rightarrow S$ be a morphism of schemes. For every n we have a natural isomorphism*

$$\Omega_{(X/S)_n/S} \simeq \rho_{n*}(\gamma_n^*(\Omega_{X/S}) \otimes \mathcal{P}_n)$$

where $\rho_n: U_n \rightarrow (X/S)_n$ and $\gamma_n: U_n \rightarrow X$ are defined in Eq. (3a), and these sheaves are isomorphic to $\rho_{n*}(\gamma_n^*(\Omega_{X/S}))$ whenever n is finite. Moreover, for $m > n$ the morphisms

$$\pi_{m,n}^*(\Omega_{(X/S)_n/S}) \rightarrow \Omega_{(X/S)_m/S}$$

induced by the truncation maps are obtained from the natural inclusion $\pi_{m,n}^*(\rho_{n*}(\mathcal{P}_n)) \rightarrow \rho_{m*}(\mathcal{P}_m)$ by tensoring with $\rho_{n*}(\gamma_n^*(\Omega_{X/S}))$, and correspond to the maps $\pi_{m,n}^*(\rho_{n*}(\mathcal{O}_{U_n})) \rightarrow \rho_{m*}(\mathcal{O}_{U_m})$ given by multiplication by t^{m-n} whenever m is finite.

Proof. Since these properties are local in X , we can assume that $X = \operatorname{Spec} A$ and $S = \operatorname{Spec} R$. Recall that all B_n -modules are regarded as A_n -modules via ρ_n^\sharp and as A -modules via γ_n^\sharp . With the same notation as in Section 4, let M be an arbitrary A_n -module. By Remark 4.3, the natural morphism

$$M \hat{\otimes}_{A_n} B_n \longrightarrow \operatorname{Hom}_{A_n}(P_n, M)$$

is an isomorphism of B_n -modules. Then, by Lemma 5.1, we have a chain of natural isomorphisms

$$\begin{aligned} \operatorname{Hom}_{A_n}(\Omega_{A_n/R}, M) &\simeq \operatorname{Der}_R(A_n, M) \\ &\simeq \operatorname{Der}_R(A, M \hat{\otimes}_{A_n} B_n) \\ &\simeq \operatorname{Hom}_A(\Omega_{A/R}, M \hat{\otimes}_{A_n} B_n) \\ &\simeq \operatorname{Hom}_A(\Omega_{A/R}, \operatorname{Hom}_{A_n}(P_n, M)) \\ &\simeq \operatorname{Hom}_{A_n}(\Omega_{A/R} \otimes_A P_n, M). \end{aligned}$$

It follows that there is a natural isomorphism of A_n -modules $\Omega_{A_n/R} \simeq \Omega_{A/R} \otimes_A P_n$. Since $\Omega_{A/R} \otimes_A P_n = (\Omega_{A/R} \otimes_A B_n) \otimes_{B_n} P_n$, this and the fact that, by Lemma 4.5, P_n is isomorphic to B_n as a B_n -module if n is finite, give the first statement. The other statements follow immediately from the second part of Lemma 5.1 and Remark 4.6. \square

6. Invariant factors and Fitting invariants

In the next section we will be interested in studying fibers of the sheaves of differentials on jet schemes $(X/S)_n$. Using Theorem 5.3, this will involve understanding the pull-back of $\Omega_{X/S}$ along a jet. As a preparation, in this section we include some remarks on these types of pull-backs.

Let X be an arbitrary scheme over base scheme S . For a given n , consider a point $\alpha_n \in (X/S)_n$, and let L_n denote the residue field of α_n . We do not assume that α_n is a closed point of $(X/S)_n$. By shrinking X around $\psi_n(\alpha_n)$, we may assume without loss of generality that $X = \operatorname{Spec} A$ and $S = \operatorname{Spec} R$ are affine. Let A_n and B_n be the algebras defined in Section 4, so that $(X/S)_n = \operatorname{Spec} A_n$ and $U_n = \operatorname{Spec} B_n$.

Algebraically, α_n is given by a morphism

$$\alpha_n^\sharp: A \longrightarrow B_n \hat{\otimes}_{A_n} L_n$$

where $B_n \hat{\otimes}_{A_n} L_n = L_n[t]/(t^{n+1})$ if n is finite and $B_\infty \hat{\otimes}_{A_\infty} L_\infty = L_\infty[[t]]$. For simplicity, in what follows we will identify $L_\infty[[t]]$ with $L_n[t]/(t^{n+1})$ when $n = \infty$, and always use the more suggestive notation $L_n[t]/(t^{n+1})$ instead of $B_n \hat{\otimes}_{A_n} L_n$.

Consider a finitely generated A -module M . We are interested in understanding the structure of its pull-back along α_n , which is given by

$$M \otimes_A L_n[t]/(t^{n+1}).$$

Notice that $L_n[t]/(t^{n+1})$ is a principal ideal ring (and a domain when $n = \infty$). Since M is finitely generated, the pull-back is also finitely generated, and the structure theory for finitely generated modules over principal ideal rings gives a unique decomposition

$$M \otimes_A L_n[t]/(t^{n+1}) \simeq (L_n[t]/(t^{n+1}))^d \oplus \bigoplus_{i \geq d} L_n[t]/(t^{e_i})$$

where $n+1 > e_d \geq e_{d+1} \geq \dots$. When $0 \leq i < d$ we set $e_i = n+1$ (so $e_i = \infty$ if $n = \infty$). If the dependency on α_n and M needs to be emphasized, we will write $d = d(\alpha_n) = d(\alpha_n, M)$ and $e_i = e_i(\alpha_n) = e_i(\alpha_n, M)$.

Definition 6.1. We call $\{e_i(\alpha_n, M)\}_{i=0}^\infty$ the sequence of *invariant factors* of M with respect to α_n . The number $d(\alpha_n, M)$ is called the *Betti number* of M with respect to α_n .

The invariant factors of M with respect to α_n determine the pull-back $M \otimes_A L[t]/(t^{n+1})$ up to isomorphism.

If α_n is the truncation of another jet α_m (so $\pi_{m,n}(\alpha_m) = \alpha_n$), the invariant factors with respect to α_n and α_m are related. We have:

$$e_i(\alpha_n) = \min\{n + 1, e_i(\alpha_m)\}.$$

Notice that the Betti numbers with respect to α_n and α_m could be different. We always have $d(\alpha_n) \geq d(\alpha_m)$.

The invariant factors are related to the Fitting ideals of M , whose definition we recall briefly. Since M is finitely generated, we can find a presentation

$$F_1 \xrightarrow{\varphi} F_0 \longrightarrow M \rightarrow 0, \quad (6a)$$

where F_0 and F_1 are free A -modules and F_0 is finitely generated. Then $\text{Fitt}^i(M) \subset A$ is the ideal generated by the minors of size $\text{rank}(F_0) - i$ of the matrix representing φ . Geometrically, a point $x \in X$ belongs to the zero locus of $\text{Fitt}^i(M)$ if and only if the dimension of the fiber $M \otimes_A k(x)$ is $> i$.

Recall that if $J \subset A$ is an ideal, $c = \text{ord}_{\alpha_n}(J)$ is defined as the number for which $\alpha_n^\sharp(J) = (t^c)$. When n is finite we pick $c \in \{0, 1, \dots, n + 1\}$, and when $n = \infty$ we have $c \in \{0, 1, \dots, \infty\}$ (we use the convention $t^\infty = 0$).

Definition 6.2. Consider the numbers

$$c_i(\alpha_n, M) := \text{ord}_{\alpha_n}(\text{Fitt}^i(M)).$$

We call $\{c_i(\alpha_n, M)\}_{i=0}^\infty$ the sequence of *Fitting invariants* of M with respect to α_n .

The Fitting invariants are determined by the invariant factors. To see this, consider the pull-back via α_n of the presentation in Eq. (6a):

$$\tilde{F}_1 \xrightarrow{\tilde{\varphi}} \tilde{F}_0 \longrightarrow M \otimes_A L_n[t]/(t^{n+1}) \rightarrow 0. \quad (6b)$$

The structure theory says that, after picking appropriate bases, $\tilde{\varphi}$ is represented by a matrix with entries $t^{e_0}, t^{e_1}, t^{e_2}, \dots$ along the main diagonal and zeros elsewhere. The minors of $\tilde{\varphi}$ are the pull-backs of the minors of φ . This property is expressed by saying that “the formation of Fitting ideals commutes with base change”, and it gives that

$$\text{Fitt}^i(M \otimes_R L_n[t]/(t^{n+1})) = \alpha_n^\sharp(\text{Fitt}^i(M)) \cdot L_n[t]/(t^{n+1}) = (t^{c_i}) \subset L_n[t]/(t^{n+1}),$$

where $c_i = \text{ord}_{\alpha_n}(\text{Fitt}^i(M))$. Here the Fitting ideals of $M \otimes_A L_n[t]/(t^{n+1})$ are computed with respect to its structure as a module over $L_n[t]/(t^{n+1})$. On the other hand, we can compute these Fitting ideals directly using the presentation in Eq. (6b). We get:

$$c_i = \min\{n + 1, e_i + e_{i+1} + e_{i+2} + \dots\}.$$

If $n = \infty$, we get that $c_i = e_i + e_{i+1} + e_{i+2} + \dots$, and we see that in this case the invariant factors are determined by the Fitting invariants. The Betti number counts the number of infinite Fitting invariants and can be interpreted geometrically as the dimension of the fiber $M \otimes_A k(\xi)$, where $\xi = \alpha_\infty(\eta)$ is the generic point of the arc α_∞ . In particular, we have that $\text{ord}_\alpha(\text{Fitt}^d(M)) < \infty$.

7. The fiber over a jet

In this section, we assume that X is a scheme of finite type over an arbitrary base scheme S . This condition on X guarantees that the sheaf of differentials $\Omega_{X/S}$ is a finitely generated \mathcal{O}_X -module. In particular, we can consider its Fitting ideals and can apply the results of Section 6.

Remark 7.1. To compute $\text{Fitt}^i(\Omega_{X/S})$, one can work locally on X and S , and use a relatively closed embedding of X in some affine space \mathbb{A}_S^N over S to get a presentation as in Eq. (6a) where φ is the Jacobian matrix of the embedding. As for any module, a point $x \in X$ belongs to the zero locus of $\text{Fitt}^i(\Omega_{X/S})$ if and only if the dimension of the fiber $\Omega_{X/S} \otimes_{\mathcal{O}_X} k(x)$ is $> i$. In particular, if X is a reduced and equidimensional scheme of finite type over a field k , then $\text{Fitt}^i(\Omega_{X/k})$ is zero when $i < \dim X$, and equals the Jacobian ideal Jac_X when $i = \dim X$.

We are interested in studying the fibers of $\Omega_{(X/S)_n/S}$, and relating them to the Fitting invariants of $\Omega_{X/S}$. We restrict ourselves to *liftable jets*. By definition, these are points in a jet scheme $(X/S)_n$ that lie in the image of the truncation map $\pi_n: (X/S)_\infty \rightarrow (X/S)_n$.

Theorem 7.2. *Let X be a scheme of finite type over a base scheme S , and consider $\alpha \in (X/S)_\infty$. We do not assume that α is a closed point of $(X/S)_\infty$. For finite n , we let $\alpha_n = \pi_n(\alpha) \in (X/S)_n$ be the truncation and L_n its residue field. Let d and $\{e_i\}$ be the Betti number and invariant factors of $\Omega_{X/S}$ with respect to α . Then, for finite $n \geq e_d$, the isomorphism $\Omega_{(X/S)_n/S} \simeq \rho_{n*}(\gamma_n^*(\Omega_{X/S}))$ given by Theorem 5.3 induces an isomorphism*

$$\Omega_{(X/S)_n/S} \otimes_{\mathcal{O}_{(X/S)_n}} L_n \simeq (L_n[t]/(t^{n+1}))^d \oplus \bigoplus_{i \geq d} L_n[t]/(t^{e_i}).$$

Proof. After restricting to a suitable open set of X , we can assume that $X = \text{Spec } A$ and $S = \text{Spec } R$. As before, we use the notation from Section 4. From Theorem 5.3, since n is finite we see that $\Omega_{A_n/S} \simeq \Omega_{A/S} \otimes_A B_n$, where B_n is considered as an A -module via $\gamma_n^\#$. This implies that

$$\Omega_{A_n/S} \otimes_{A_n} L_n \simeq \Omega_{A/S} \otimes_A L_n[t]/(t^{n+1}),$$

where $L_n[t]/(t^{n+1})$ is considered as an A -module via $\alpha_n^\#$. The condition $n \geq e_d$ guarantees that Betti numbers of $\Omega_{A/S}$ with respect to α and α_n coincide. Moreover, starting at the position $i = d$ the invariant factors also coincide. The theorem now follows from the definition of the invariant factors with respect to the jet α_n . \square

From this theorem it is immediate to compute the dimensions of the fibers of $\Omega_{(X/S)_n/S}$ over liftable jets. We recall that the Betti number of $\Omega_{X/S}$ with respect to the arc α can be interpreted as the dimension of the fiber $\Omega_{X/S} \otimes_{\mathcal{O}_X} k(\xi)$, where $\xi = \alpha(\eta) \in X$ is the generic point of α .

Corollary 7.3. *With the same assumptions as Theorem 7.2, consider the ideal sheaf $\mathcal{J}_d := \text{Fitt}^d(\Omega_{X/S})$. Then, for finite $n \geq \text{ord}_\alpha(\mathcal{J}_d)$ we have*

$$\dim_{L_n} (\Omega_{(X/S)_n/S} \otimes_{\mathcal{O}_{(X/S)_n}} L_n) = (n+1)d + \text{ord}_\alpha(\mathcal{J}_d).$$

Proof. Since $\text{ord}_\alpha(\mathcal{J}_d) = c_d = e_d + e_{d+1} + \dots \geq e_d$, the result follows immediately from Theorem 7.2. \square

The next corollary recovers [dFD14, Proposition 5.1].

Corollary 7.4. *In addition to the assumptions of Theorem 7.2, assume that $S = \operatorname{Spec} k$ for a field k , that X is reduced and equidimensional over k , and that the arc α is not completely contained in the singular locus of X . Then, for finite $n \geq \operatorname{ord}_\alpha(\operatorname{Jac}_X)$ we have*

$$\dim_{L_n}(\Omega_{X_n/k} \otimes_{\mathcal{O}_{X_n}} L_n) = (n+1) \dim X + \operatorname{ord}_\alpha(\operatorname{Jac}_X).$$

Proof. With the additional assumptions, we see that the Betti number of $\Omega_{X/S}$ is with respect to α is $d = \dim X$, and therefore $\mathcal{J}_d = \operatorname{Fitt}^d(\Omega_{A/k}) = \operatorname{Jac}_X$. The result is just a restatement of Corollary 7.3 in this case. \square

8. Embedding dimension

We now study embedding dimensions of arcs and jets. Starting with this section and for the remainder of the paper, we assume that X is a scheme of finite type over a perfect field k .

In the following, let $\alpha \in X_\infty$ be a point, and denote by $L = L_\infty$ the residue field of α . We do not assume that α is a closed point of X_∞ . For finite n , we let $\alpha_n = \pi_n(\alpha)$ be the truncations, and denote by L_n their residue fields. It will be convenient to also allow the notation α_∞ for α .

For n finite, we denote $\dim(\alpha_n) := \operatorname{tr. deg}(L_n/k)$. Since the ground field k is assumed to be perfect, we have

$$\dim(\alpha_n) = \dim_{L_n}(\Omega_{L_n/k}).$$

We start with some preliminary lemmas. For ease of notation, in the discussion of these preliminary properties we restrict ourselves to the affine setting and assume that $X = \operatorname{Spec} A$ where A is a finitely generated k -algebra. We apply the notation from Section 4 with $R = k$.

For each n , we let $I_n \subset A_n$ be the prime ideal defining α_n . When $m > n$ we have inclusions $I_n \subset I_m$. The Zariski tangent space of X_n at α_n is the dual of the L_n -vector space I_n/I_n^2 , and hence the embedding dimension of X_n at α_n is given by

$$\operatorname{emb. dim}(\mathcal{O}_{X_n, \alpha_n}) = \dim_{L_n}(I_n/I_n^2).$$

Note that there are natural maps $I_n/I_n^2 \rightarrow I_m/I_m^2$ whenever $m > n$, and

$$I_\infty/I_\infty^2 = \operatorname{inj} \lim_{n \rightarrow \infty} (I_n/I_n^2).$$

Lemma 8.1. *With notation as above, let d be the Betti number of $\Omega_{A/k}$ with respect to the arc α , and consider the ideal $J_d := \operatorname{Fitt}^d(\Omega_{A/k})$. Then, for finite $n \geq \operatorname{ord}_\alpha(J_d)$ we have*

$$\operatorname{emb. dim}(\mathcal{O}_{X_n, \alpha_n}) = (n+1)d - \dim(\alpha_n) + \operatorname{ord}_\alpha(J_d).$$

Proof. Applying [Mat89, Theorem 25.2] to the sequence $k \rightarrow (A_n)_{I_n} \rightarrow L_n$, we get an exact sequence

$$0 \rightarrow I_n/I_n^2 \rightarrow \Omega_{A_n/k} \otimes_{A_n} L_n \rightarrow \Omega_{L_n/k} \rightarrow 0.$$

Here we used the assumption that k is perfect. The lemma now follows from Corollary 7.3 and the equality $\dim(\alpha_n) = \dim_{L_n}(\Omega_{L_n/k})$. \square

Lemma 8.2. *With the same assumptions as Lemma 8.1, let $\Omega_{A_n/k} \otimes_{A_n} A_m \rightarrow \Omega_{A_m/k}$ be the map induced by the truncation morphism $\pi_{m,n}: X_m \rightarrow X_n$. Then, for finite $n \geq \operatorname{ord}_\alpha(J_d)$ and $m \geq n + \operatorname{ord}_\alpha(J_d)$, we have*

$$K := \ker(\Omega_{A_n/k} \otimes_{A_n} L \rightarrow \Omega_{A_m/k} \otimes_{A_m} L) \simeq \bigoplus_{i \geq d} L[t]/(t^{e_i}).$$

In particular, $\dim_L(K) = \text{ord}_\alpha(J_d)$.

Proof. By Theorems 5.3 and 7.2 (see also Remark 4.6), we see that K is isomorphic to the kernel of the map

$$\left((L[t]/(t^{n+1}))^d \oplus \bigoplus_{i \geq d} L[t]/(t^{e_i}) \right) \xrightarrow{\cdot t^{m-n}} \left((L[t]/(t^{m+1}))^d \oplus \bigoplus_{i \geq d} L[t]/(t^{e_i}) \right)$$

given by multiplication by t^{m-n} . Since we have $m - n \geq \text{ord}_\alpha(J_d) \geq e_i$ for all $i \geq d$, we see that $K \simeq \bigoplus_{i \geq d} L[t]/(t^{e_i})$. The last assertion follows from the fact that $\sum_{i \geq d} e_i = \text{ord}_\alpha(J_d)$. \square

Lemma 8.3. *With the same assumptions as Lemma 8.1, consider the natural morphism*

$$\lambda_n: I_n/I_n^2 \otimes_{A_n} L \longrightarrow I_\infty/I_\infty^2$$

induced by the truncation map. Then for finite $n \geq \text{ord}_\alpha(J_d)$, we have

$$\dim_L(\text{Im}(\lambda_n)) = (n+1)d - \dim(\alpha_n).$$

Proof. Consider $m \geq n + \text{ord}_\alpha(J_d)$. We have the following commutative diagram with exact rows and columns:

$$\begin{array}{ccccccc} & & 0 & & 0 & & \\ & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & K & \xlongequal{\quad} & K & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & I_n/I_n^2 \otimes L & \longrightarrow & \Omega_{A_n/k} \otimes L & \longrightarrow & \Omega_{L_n/k} \otimes L \longrightarrow 0 \\ & & \downarrow \lambda_{n,m} & & \downarrow & & \downarrow \\ 0 & \longrightarrow & I_m/I_m^2 \otimes L & \longrightarrow & \Omega_{A_m/k} \otimes L & \longrightarrow & \Omega_{L_m/k} \otimes L \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & Q & \longrightarrow & C & \longrightarrow & \Omega_{L_m/L_n} \otimes L \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ & & 0 & & 0 & & 0 \end{array}$$

From the first column of the diagram and Lemmas 8.1 and 8.2, we see that

$$\dim_L(\text{Im}(\lambda_{n,m})) = \dim_L(I_n/I_n^2 \otimes L) - \dim_L(K) = (n+1)d - \dim(\alpha_n).$$

Since $\text{Im}(\lambda_n) = \text{inj} \lim_m \text{Im}(\lambda_{n,m})$, the result follows. \square

Lemma 8.4. *With the same assumptions as Lemma 8.1, we have that the numerical sequence $\{(n+1)d - \dim(\alpha_n)\}$ is eventually non-decreasing. Moreover:*

$$\text{emb. dim}(\mathcal{O}_{X_\infty, \alpha}) = \lim_{n \rightarrow \infty} ((n+1)d - \dim(\alpha_n)).$$

Proof. Consider the maps λ_n of Lemma 8.3. Since $I_\infty/I_\infty^2 = \text{inj} \lim(I_n/I_n^2)$, we also have that $I_\infty/I_\infty^2 = \text{inj} \lim(\text{Im}(\lambda_n))$. Therefore the assertions follow from Lemma 8.3 and the fact that $\text{Im}(\lambda_n) \subset \text{Im}(\lambda_{n+1})$. \square

We now return to the global case of schemes of finite type over k .

Theorem 8.5. *Assume that X is a reduced and equidimensional scheme of finite type over k . Consider an arc $\alpha \in X_\infty$ that is not completely contained in the singular locus of X , and let $\alpha_n = \pi_n(\alpha)$ be its truncations. Then we have*

$$\text{emb. dim}(\mathcal{O}_{X_\infty, \alpha}) = \text{emb. dim}(\mathcal{O}_{X_n, \alpha_n}) - \text{ord}_\alpha(\text{Jac}_X)$$

for all sufficiently large integers n .

Proof. By Lemmas 8.1 and 8.4. □

Implicit in the works on motivic integration is the definition of codimension of a constructible subset of the arc space of a smooth variety. This was formalized and extended to singular varieties in [ELM04, dFEI08, dFM15].

Definition 8.6. Given an arc $\alpha \in X_\infty$, consider its generic point $\xi = \alpha(\eta) \in X$. Recall that the dimension of X at ξ , denoted $\dim_\xi(X)$, is defined as the infimum of the dimensions of all open neighborhoods of ξ in X . For the truncations $\alpha_n = \pi_n(\alpha)$ we define their *expected codimension* as

$$\text{exp. codim}(\alpha_n, X_n) = (n+1) \dim_\xi(X) - \dim(\alpha_n).$$

The *jet codimension* of α in X_∞ is defined to be

$$\text{jet. codim}(\alpha, X_\infty) = \lim_{n \rightarrow \infty} \text{exp. codim}(\alpha_n, X_n).$$

Remark 8.7. Lemma 8.4 guarantees that the limit in the definition of jet codimension always exists. This fact is also proved in [dFM15, Lemma 4.13].

Remark 8.8. The expected codimension is the codimension of α_n in X_n when X is smooth, or more generally when X is reduced and X_n is equidimensional. If X is reduced and equidimensional and k is a field of characteristic 0, then every irreducible component of X_∞ dominates an irreducible component of X by Kolchin's Irreducibility Theorem [Kol73]. It follows that the set of liftable jets $\pi_n(X_\infty) \subset X_n$ has pure dimension $(n+1) \dim X$, and therefore one can think of the expected codimension of α_n as its codimension in this set.

For the next theorem, recall that since the ground field is perfect, a point x on a scheme of finite type X is singular if and only if $\dim(\Omega_X \otimes k(x)) > \dim_x(X)$. In particular, non-reduced points of X are singular. We denote by $\text{Sing } X$ the singular locus of X .

Theorem 8.9. *Let X be scheme of finite type and let X_{red} be its reduction. Consider $\alpha \in X_\infty$, and let $\xi = \alpha(\eta) \in X$ be its generic point.*

(1) *For every α , we have*

$$\text{jet. codim}(\alpha, X_\infty) = \text{jet. codim}(\alpha, (X_{\text{red}})_\infty) = \text{emb. dim}(\mathcal{O}_{(X_{\text{red}})_\infty, \alpha}).$$

(2) *If $\alpha \in (\text{Sing } X)_\infty$, then*

$$\text{emb. dim}(\mathcal{O}_{X_\infty, \alpha}) = \infty.$$

(3) *If $\alpha \in Y_\infty$ where $Y \subset X$ is a closed subscheme with $\dim_\xi(Y) < \dim_\xi(X)$, then*

$$\text{jet. codim}(\alpha, X_\infty) = \text{emb. dim}(\mathcal{O}_{X_\infty, \alpha}) = \infty.$$

Proof. We first prove (3). We can assume without loss of generality that Y is reduced. Since the ground field is perfect, we can also assume that the generic point of α is a smooth point on Y . By the geometric interpretation of Betti numbers, we see that

$$d(\alpha, \Omega_Y) = \dim_\xi(Y) < \dim_\xi(X) \leq d(\alpha, \Omega_X).$$

Applying Lemma 8.4 to Y and the definition of jet codimension, we see that

$$\text{jet. codim}(\alpha, X_\infty) = \text{emb. dim}(\mathcal{O}_{Y_\infty, \alpha}) + \lim_{n \rightarrow \infty} (n+1)(\dim_\xi(X) - \dim_\xi(Y)) = \infty,$$

and applying Lemma 8.4 to both X and Y we see that

$$\text{emb. dim}(\mathcal{O}_{X_\infty, \alpha}) = \text{emb. dim}(\mathcal{O}_{Y_\infty, \alpha}) + \lim_{n \rightarrow \infty} (n+1)(d(\alpha, \Omega_X) - \dim_\xi(Y)) = \infty.$$

This proves (3).

Next, we prove (1). First notice that the definition of jet codimension does not detect non-reduced structure, so clearly $\text{jet. codim}(\alpha, X_\infty) = \text{jet. codim}(\alpha, (X_{\text{red}})_\infty)$. We can therefore assume that X is a reduced scheme, in which case we see that the identity $\text{jet. codim}(\alpha, X_\infty) = \text{emb. dim}(\mathcal{O}_{Y_\infty, \alpha})$ holds by Lemma 8.4 if $\alpha \notin (\text{Sing } X)_\infty$, and by (3) if $\alpha \in (\text{Sing } X)_\infty$.

Finally, to prove (2), assume that $\alpha \in (\text{Sing } X)_\infty$. If $\dim_\xi(\text{Sing } X)_\infty < \dim_\xi X$, then the assertion follows by (3). Otherwise, X is non-reduced at ξ , which implies that $d = d(\alpha, \Omega_X) > \dim X$, and hence that $\text{jet. codim}(\alpha, X_\infty) = \infty$ by Lemma 8.4. \square

Corollary 8.10. *If X is a reduced scheme of finite type, then*

$$\text{jet. codim}(\alpha, X_\infty) = \text{emb. dim}(\mathcal{O}_{X_\infty, \alpha})$$

for every $\alpha \in X_\infty$.

9. The birational transformation rule

We study now how birational morphisms affect the embedding dimension of arcs.

In this section all schemes will be reduced and of finite type over a perfect field k . Given any two such schemes X and Y , we will say that a morphism $f: Y \rightarrow X$ is *birational over a union of components* if there exist a dense open set $V \subset Y$ and a (not necessarily dense) open set $U \subset X$ such that $f(V) \subset U$ and the restriction $f|_V: V \rightarrow U$ is an isomorphism. If U is dense in X , or equivalently if f is dominant, then we say that f is *birational*.

If $f: Y \rightarrow X$ is birational over a union of components, then the sheaf of relative differentials $\Omega_{Y/X}$ is torsion. We define $\text{Jac}_f = \text{Fitt}^0(\Omega_{Y/X})$, and call it the *Jacobian ideal* of f . The zero locus of Jac_f is called the *exceptional locus* of f .

Lemma 9.1. *Let X and Y be reduced schemes of finite type over a perfect field, and consider a proper map $f: Y \rightarrow X$ that is birational over a union of components. Let $\beta \in Y_\infty$ and consider $\alpha = f_\infty(\beta) \in X_\infty$. If β is not completely contained in the exceptional locus of f , then the residue fields of α and β are equal.*

Proof. Let L and K be the residue fields of α and β , respectively. Since $\alpha = f_\infty(\beta)$, we have $L \subset K$. Consider α as map $\alpha: \text{Spec } L[[t]] \rightarrow X$. The hypothesis on β guarantees that the generic point $\alpha(\eta)$ of α lies in the locus over which f is an isomorphism, and therefore it can be lifted. The valuative criterion of proneness gives a unique lift of α to an arc $\tilde{\alpha}: \text{Spec } L[[t]] \rightarrow Y$. This corresponds to a morphism $\text{Spec } L \rightarrow Y_\infty$ whose image is β by construction. This implies that $K \subset L$, as required. \square

Theorem 9.2. *Let X and Y be reduced schemes of finite type over a perfect field. Consider a proper map $f: Y \rightarrow X$ that is birational over a union of components. Let $\beta \in Y_\infty$ and consider $\alpha = f_\infty(\beta) \in X_\infty$. Assume that Y is smooth at $\beta(0)$. Then*

$$\text{emb. dim}(\mathcal{O}_{X_\infty, \alpha}) = \text{emb. dim}(\mathcal{O}_{Y_\infty, \beta}) + \text{ord}_\beta(\text{Jac}_f).$$

In view of Corollary 8.10, the above equality is equivalent to

$$\text{jet.codim}(\alpha, X_\infty) = \text{jet.codim}(\beta, Y_\infty) + \text{ord}_\beta(\text{Jac}_f).$$

Notice that several of these numbers could be infinite. For example, if β has infinite embedding dimension, the theorem implies that α also has infinite embedding dimension. Conversely, if α has infinite embedding dimension and β is not completely contained in the exceptional locus, then β has infinite embedding dimension.

Proof of Theorem 9.2. If β is contained in the exceptional locus, then α is contained in the image of the exceptional locus, and hence both embedding dimensions are infinite by Corollary 8.10. Thus we assume that β is not contained in the exceptional locus. By Lemma 9.1 both α and β have the same residue field, which we call L . Let I and J be the ideals defining α and β . Since the ground field is perfect, we have the following diagram:

$$\begin{array}{ccccccc} 0 & \longrightarrow & I/I^2 & \longrightarrow & \Omega_{X_\infty/k} \otimes_{\mathcal{O}_{X_\infty}} L & \longrightarrow & \Omega_{L/k} \longrightarrow 0 \\ & & \downarrow & & \downarrow \varphi & & \parallel \\ 0 & \longrightarrow & J/J^2 & \longrightarrow & \Omega_{Y_\infty/k} \otimes_{\mathcal{O}_{Y_\infty}} L & \longrightarrow & \Omega_{L/k} \longrightarrow 0 \end{array}$$

Therefore, the theorem will follow if we show that

$$\dim_L(\ker \varphi) - \dim_L(\text{coker } \varphi) = \text{ord}_\beta(\text{Jac}_f).$$

Recall the universal arc $\rho_\infty: U_\infty \rightarrow X_\infty$ and the sheaf \mathcal{P}_∞ on U_∞ defined in Section 4. For ease of notation, we denote

$$B_L := L[[t]] = \rho_{\infty*}(\mathcal{O}_{U_\infty}) \otimes_{\mathcal{O}_{X_\infty}} L$$

and

$$P_L := L((t))/tL[[t]] = \rho_{\infty*}(\mathcal{P}_\infty) \otimes_{\mathcal{O}_{X_\infty}} L.$$

We can regard B_L both as an \mathcal{O}_X -algebra via the arc α and as an \mathcal{O}_Y -algebra via β . Then P_L , which is naturally a B_L -module, becomes both an \mathcal{O}_X -module and an \mathcal{O}_Y -module.

The map f induces a natural sequence of sheaves of differentials:

$$\Omega_{X/k} \otimes_{\mathcal{O}_X} \mathcal{O}_Y \xrightarrow{f^*} \Omega_{Y/k} \longrightarrow \Omega_{Y/X} \longrightarrow 0.$$

After pulling back to the arcs α and β we get:

$$\Omega_{X/k} \otimes_{\mathcal{O}_X} B_L \xrightarrow{\psi} \Omega_{Y/k} \otimes_{\mathcal{O}_Y} B_L \longrightarrow \Omega_{Y/X} \otimes_{\mathcal{O}_Y} B_L \longrightarrow 0.$$

All of the terms in this sequence are finitely generated modules over $B_L = L[[t]]$, and therefore they are direct sums of cyclic modules. In particular, ψ can be represented by a matrix with entries t^{a_0}, t^{a_1}, \dots along the main diagonal and zeroes elsewhere. In the language of Section 6, the a_i can be chosen to be the invariant factors of the module $\Omega_{Y/X}$ with respect to the arc β . In particular, we have that $\text{ord}_\beta(\text{Jac}_f) = \sum_{i \geq 0} a_i$. Since β is not contained in the exceptional locus, we have $a_i < \infty$ for all i .

Since Y is smooth at $\beta(0)$, the middle term $\Omega_{Y/k} \otimes_{\mathcal{O}_Y} B_L$ is free. Write $\Omega_{X/k} \otimes_{\mathcal{O}_X} B_L = F \oplus T$, where F is free and T is torsion. Then ψ annihilates T and the matrix representing $\psi|_F$ contains all the diagonal entries t^{a_0}, t^{a_1}, \dots .

Theorem 5.3 says that φ is obtained from ψ by tensoring with P_L :

$$\begin{array}{ccccccc} 0 & \longrightarrow & K & \longrightarrow & \Omega_{X/k} \otimes_{\mathcal{O}_X} P_L & \xrightarrow{\varphi} & \Omega_{Y/k} \otimes_{\mathcal{O}_Y} P_L \longrightarrow \Omega_{Y/X} \otimes_{\mathcal{O}_Y} P_L \longrightarrow 0 \\ & & & & \parallel & & \parallel \\ & & & & \Omega_{X_\infty/k} \otimes_{\mathcal{O}_{X_\infty}} L & & \Omega_{Y_\infty/k} \otimes_{\mathcal{O}_{Y_\infty}} L \end{array}$$

Notice that P_L is a divisible B_L -module, and hence tensoring with P_L kills torsion. Since $\Omega_{Y/X} \otimes_{\mathcal{O}_Y} B_L$ is torsion (because β is not contained in the exceptional locus), we see that φ is surjective. Moreover $\Omega_{X/k} \otimes_{\mathcal{O}_X} P_L = F \otimes_{B_L} P_L$, and φ is represented by the same matrix as $\psi|_F$. We get that $K = \bigoplus_{i \geq 0} K_i$, where K_i is the kernel of the map $P_L \rightarrow P_L$ given by multiplication by t^{a_i} . An easy computation shows that $\dim_L(K_i) = a_i$. Therefore $\dim_L(K) = \sum_{i \geq 0} a_i = \text{ord}_\beta(\text{Jac}_f)$, and the result follows. \square

Theorem 9.3. *Let X and Y be reduced schemes of finite type over a perfect field. Consider a proper map $f: Y \rightarrow X$ that is birational over a union of components. Let $\beta \in Y_\infty$ and consider $\alpha = f_\infty(\beta) \in X_\infty$. Then*

$$\text{emb. dim}(\mathcal{O}_{Y_\infty, \beta}) \leq \text{emb. dim}(\mathcal{O}_{X_\infty, \alpha}) \leq \text{emb. dim}(\mathcal{O}_{Y_\infty, \beta}) + \text{ord}_\beta(\text{Jac}_f).$$

Proof. The proof is almost identical to the one of Theorem 9.2. Using the notation of that theorem, the main difference is that $\Omega_{Y/k} \otimes_{\mathcal{O}_Y} B_L$ is no longer a free B_L -module. In particular some of the invariant factors t^{a_i} might correspond to maps from torsion factors of $\Omega_{X/k} \otimes_{\mathcal{O}_X} B_L$ to torsion factors of $\Omega_{Y/k} \otimes_{\mathcal{O}_Y} B_L$. After we tensor with P_L , these “torsion contributions” get killed. We see that the dimension of $K = \ker \varphi$ is the sum of the invariant factors not corresponding to the torsion factors. In any case, we get that $\dim_L(K) \leq \text{ord}_\beta(\text{Jac}_f)$, and the result follows. \square

Corollary 9.4. *Let $f: Y \rightarrow X$ be a proper birational morphism between reduced schemes of finite type over a perfect field. Then the induced map $f_\infty: Y_\infty \rightarrow X_\infty$ induces a bijection*

$$\{\beta \in Y_\infty \mid \text{emb. dim}(\mathcal{O}_{Y_\infty, \beta}) < \infty\} \xrightarrow{1-1} \{\alpha \in X_\infty \mid \text{emb. dim}(\mathcal{O}_{X_\infty, \alpha}) < \infty\}.$$

Proof. Theorem 9.3 implies that Y_∞ has finite embedding dimension at a point β if and only if X_∞ has finite embedding dimension at $f_\infty(\beta)$. To conclude, it suffices to observe that if $\alpha \in X_\infty$ is not in the image of f_∞ then X_∞ has infinite embedding dimension at α . Indeed, by the valuative criterion of properness, α must be fully contained in the indeterminacy locus of $f^{-1}: X \dashrightarrow Y$. Since f is a birational map of reduced schemes, the indeterminacy locus of f^{-1} has dimension strictly smaller than the dimension of X at any of its points, and therefore we have $\text{emb. dim}(\mathcal{O}_{X_\infty, \alpha}) = \infty$ by Theorem 8.9. \square

Theorem 9.3 also implies the following basic property.

Corollary 9.5. *Let X be a reduced scheme of finite type and $Y \subset X$ a union of irreducible components of X . For every $\alpha \in Y_\infty$, we have*

$$\text{emb. dim}(\mathcal{O}_{Y_\infty, \alpha}) = \text{emb. dim}(\mathcal{O}_{X_\infty, \alpha}).$$

Proof. Since $f: Y \rightarrow X$ is injective, we have $\text{Jac}_f = \mathcal{O}_Y$ and hence the assertion follows directly from Theorem 9.3. \square

Remark 9.6. A more direct proof of this property relies on the observation that if α is fully contained in the intersection of Y with the union of the other irreducible components of X then both local rings $\mathcal{O}_{X_\infty, \alpha}$ and $\mathcal{O}_{Y_\infty, \alpha}$ have infinite embedding dimension by Theorem 8.9, and otherwise there is an isomorphism $\mathcal{O}_{Y_\infty, \alpha} \cong \mathcal{O}_{X_\infty, \alpha}$.

10. Maximal divisorial arcs

In this section we study arcs that are naturally associated with divisorial valuations. As in the previous section, we let X be a reduced scheme of finite type over a perfect field k .

Definition 10.1. A *valuation* on X is intended to be a k -trivial valuation of the function field of one of the irreducible components of X with center in X . A *divisorial valuation* on X is a valuation v of the form $v = q \operatorname{ord}_E$ where q is a positive integer and E is a prime divisor on a normal scheme Y with a morphism $f: Y \rightarrow X$ that is birational over a union of irreducible components of X . For a divisorial valuation v , the number $\widehat{k}_v(X) := v(\operatorname{Jac}_f)$ depends only on v (not on the particular map f), and is called the *Mather discrepancy* of v over X . When $v = \operatorname{ord}_E$ (so $q = 1$), we write $\widehat{k}_E(X)$.

If we denote by \mathcal{K}_X the sheaf of rational functions of X in the sense of [Kle79], that is, $\mathcal{K}_X = \bigoplus_{\eta} \mathcal{O}_{X,\eta}$ where η ranges among the generic points of the irreducible components of X , then a valuation of X can be thought as a function $v: \mathcal{K}_X \rightarrow (-\infty, \infty]$ which restricts to a Krull valuation on one of the summands $\mathcal{O}_{X,\eta}$ and is constant equal to ∞ on the other summands. If $v = q \operatorname{ord}_E$ is a divisorial valuation on X , then E is a divisor on one of the irreducible components of Y , and the component of X dominated by it corresponds to the summand of \mathcal{K}_X where the valuation is non-trivial.

Definition 10.2. A point $\alpha \in X_{\infty}$ is a *maximal divisorial arc* if $\operatorname{ord}_{\alpha}$ extends to a divisorial valuation on X and α is maximal among all points $\gamma \in X_{\infty}$ with $\operatorname{ord}_{\gamma} = \operatorname{ord}_{\alpha}$ (that is, α is not the specialization of any other such point γ).

In general, for an arc $\alpha \in X_{\infty}$, the function $\operatorname{ord}_{\alpha}$ is only defined on $\mathcal{O}_{X,\alpha(0)}$. If α is a maximal divisorial arc, then we write $\operatorname{ord}_{\alpha} = q \operatorname{ord}_E$ and think of it as a function on \mathcal{K}_X . Note that for other arcs β (for instance, if β is contained in Y_{∞} for a smaller dimensional scheme $Y \subset X$) there may not be a natural way to extend $\operatorname{ord}_{\beta}$ to \mathcal{K}_X .

Let $f: Y \rightarrow X$ be a proper morphism from a normal scheme Y that is birational over a union of components. Let $E \subset Y$ be a prime divisor, and let $E^{\circ} \subset E$ be the open set where both Y and E are smooth and none of the other components of the exceptional locus of f intersect E . For any positive integer q , consider the *contact set*

$$\operatorname{Cont}^{\geq q}(E^{\circ}, Y) \subset Y_{\infty},$$

which is defined to be the set of arcs in Y with order of contact at least q with E at a point in E° . Since Y is smooth along E° , the truncations $Y_m \rightarrow Y_n$ are affine bundles over an open set containing E° , and this implies that $\operatorname{Cont}^{\geq q}(E^{\circ}, Y)$ is irreducible.

Lemma 10.3. *With the above notation, the image under $f_{\infty}: Y_{\infty} \rightarrow X_{\infty}$ of the generic point of $\operatorname{Cont}^{\geq q}(E^{\circ}, Y)$ is a maximal divisorial arc on X , and any such arc arises in this way.*

Proof. Let β be the generic point of $\operatorname{Cont}^{\geq q}(E^{\circ}, Y)$ and $\alpha = f_{\infty}(\beta)$. It is elementary to see that $\operatorname{ord}_{\beta} = q \operatorname{ord}_E$. By the definition of f_{∞} , we have $\operatorname{ord}_{\alpha} = \operatorname{ord}_{\beta}$, and therefore $\operatorname{ord}_{\alpha} = q \operatorname{ord}_E$.

We may assume without loss of generality that f is dominant (this is not essential, but it makes the wording of the proof more clear). If $\gamma \in X$ is any arc with $\operatorname{ord}_{\gamma} = \operatorname{ord}_{\alpha}$, then γ cannot be fully contained in the indeterminacy locus of f^{-1} , and therefore it lifts to an arc $\tilde{\gamma}$ on Y by the valuative criterion of properness. Since $\operatorname{ord}_{\tilde{\gamma}} = q \operatorname{ord}_E$, we see that $\tilde{\gamma}$ must dominate the generic point of E and hence lie in $\operatorname{Cont}^{\geq q}(E^{\circ}, Y)$. It

follows that γ is a specialization of α , and therefore α is a maximal divisorial arc. This argument also shows that any maximal divisorial arc arises in this way. \square

Theorem 10.4. *Let X be a reduced scheme of finite type over a perfect field. For every divisorial valuation $q \operatorname{ord}_E$ on X there exists a unique maximal divisorial arc $\alpha \in X_\infty$ with $\operatorname{ord}_\alpha = q \operatorname{ord}_E$. Moreover:*

$$\operatorname{emb. dim}(\mathcal{O}_{X_\infty, \alpha}) = q(\widehat{k}_E(X) + 1).$$

Proof. The first assertion is a direct consequence of Lemma 10.3. The formula for the embedding dimension follows from Theorem 9.2, after we notice that if β is the generic point of $\operatorname{Cont}^{\geq q}(E^\circ, Y)$ then $\operatorname{emb. dim}(\mathcal{O}_{Y_\infty, \beta}) = q$, which is an easy computation given that Y is smooth. \square

Since by Corollary 8.10 we have $\operatorname{jet. codim}(\alpha) = \operatorname{emb. dim}(\mathcal{O}_{X_\infty, \alpha})$, we obtain the next corollary which recovers the formula in [dFEI08, Theorem 3.8].

Corollary 10.5. *With the same assumptions as in Theorem 10.4, we have*

$$\operatorname{jet. codim}(\alpha, X_\infty) = q(\widehat{k}_E(X) + 1).$$

The following related result has been recently proved, by different methods, by Mourtada and Reguera.

Theorem 10.6 ([Reg, MR]). *Let X be a variety defined over a field of characteristic 0 and $\alpha \in X_\infty$ is a maximal divisorial point corresponding to a valuation $q \operatorname{ord}_E$. Then*

$$\operatorname{emb. dim}(\widehat{\mathcal{O}_{X_\infty, \alpha}}) = \operatorname{emb. dim}(\mathcal{O}_{(X_\infty)_{\operatorname{red}}, \alpha}) = q(\widehat{k}_E(X) + 1).$$

It is interesting to compare Theorems 10.4 and 10.6. In general, $\mathcal{O}_{X_\infty, \alpha}$ is not necessary a Noetherian ring (see [Reg09, Example 3.16]), and therefore neither result implies the other. Instead, we learn something new by comparing the two results, namely, that in characteristic 0 the local ring of X_∞ at a maximal divisorial arc has the same embedding dimension as its completion and its reduction.

11. Stable points

Throughout this section, let X be a scheme of finite type over a perfect field k . For an arc $\alpha \in X_\infty$, we denote by $\alpha_n = \pi_n(\alpha) \in X_n$ its truncations.

The goal of this section is to characterize local rings $\mathcal{O}_{X_\infty, \alpha}$ of finite embedding dimension. To this end, we recall the following definition.

Definition 11.1. A point $\alpha \in X_\infty$ is said to be *stable* if α is the generic point of an irreducible constructible subset of X_∞ .

This definition requires some comments. The definition of constructible set in X_∞ is here intended in the sense of [EGA III₁]. That is, a subset of X_∞ is *constructible* if and only if it is a finite union of finite intersections of retrocompact open sets and their complements, where a subset $Z \subset X_\infty$ is said to be *retrocompact* if for every quasi-compact open set $U \subset X_\infty$, the intersection $Z \cap U$ is quasi-compact. It is a general fact that a subset of X_∞ is constructible if and only if it is the (reduced) inverse image of a constructible subset of X_n for some finite n [EGA IV₃, Théorème (8.3.11)]. Constructible sets in arc spaces are also known as *cylinders* (e.g., see [ELM04, EM09]). A constructible set is said to be *irreducible* if its closure is irreducible, in which case, by definition, its *generic point* is the generic point of its closure.

Stable points of arc spaces were first studied by Denef and Loeser [DL99] and later by Reguera [Reg06, Reg09]. In order to relate stable points to the embedding dimension of their local rings, we rely on the following technical lemma of Denef and Loeser.

Lemma 11.2 ([DL99, EM09]). *Assume that X is a variety, and let e be a non-negative integer. Then, for every $n \geq e$, the natural projection $\pi_{n+1}(X_\infty) \rightarrow \pi_n(X_\infty)$ induces a piecewise trivial fibration*

$$\tau_n: \pi_{n+1}(\text{Cont}^e(\text{Jac}_X)) \rightarrow \pi_n(\text{Cont}^e(\text{Jac}_X))$$

with fiber $\mathbb{A}^{\dim(X)}$, where $\text{Cont}^e(\text{Jac}_X) \subset X_\infty$ is the set of arcs whose order of contact with Jac_X is equal to e .

Remark 11.3. This property was first proved (in a weaker form, in characteristic 0) in [DL99, Lemma 4.1]. The stronger statement given here is taken from [EM09, Proposition 4.1]. There, the ground field k is assumed to be algebraically closed, but the proof still holds if k is only assumed to be perfect. Even though this is not explicitly stated in [EM09], the proof also shows that the stratification making τ_n a piecewise trivial fibration can be chosen compatibly as n varies, so that if $\Sigma \subset \pi_n(\text{Cont}^e(\text{Jac}_X))$ is contained in a stratum over which τ_n is a trivial fibration, then $\tau_n^{-1}(\Sigma) \subset \pi_{n+1}(\text{Cont}^e(\text{Jac}_X))$ is contained in a stratum over which τ_{n+1} is a trivial fibration.

Lemma 11.4. *Let X be a variety and $\alpha \in X_\infty$ be an arc with $e = \text{ord}_\alpha(\text{Jac}_X) < \infty$. For $n \geq e$, let $\tau_n: \pi_{n+1}(\text{Cont}^e(\text{Jac}_X)) \rightarrow \pi_n(\text{Cont}^e(\text{Jac}_X))$ be the map defined in Lemma 11.2. Then the following are equivalent:*

- (1) α is a stable point;
- (2) there exists an integer $m \geq e$ such that for every $n \geq m$ the fiber $\pi_n^{-1}(\alpha_n)$ is irreducible with generic point α ;
- (3) there exists an integer $m \geq e$ such that for every $n \geq m$ the fiber $\tau_n^{-1}(\alpha_n)$ is irreducible with generic point α_{n+1} ;
- (4) $\text{jet.codim}(\alpha) < \infty$.

Proof. Let α be the generic point of an irreducible constructible set W . We can fix an integer m such that, for every $n \geq m$, there exists a constructible set $W_n \subset X_n$ such that $W = \pi_n^{-1}(W_n)$. After replacing W_n with $W_n \cap \overline{\pi_n(W)}$, we can assume that W dominates W_n and the latter is irreducible. Then α_n is the generic point of W_n and hence α is the generic point of $\pi_n^{-1}(\alpha_n)$. This shows that (1) \Rightarrow (2).

The implication (2) \Rightarrow (3) is clear from Lemma 11.2.

Assume now that (3) holds. Let $\Sigma \subset \pi_m(\text{Cont}^e(\text{Jac}_X))$ be the intersection of the closure of α_m in X_m with a stratum over which τ_m is a trivial fibration, as in Lemma 11.2. Then we see by Lemma 11.2 and Remark 11.3 that $W = \pi_m^{-1}(\Sigma)$ is an irreducible constructible set and α is its generic point. This shows (1).

To conclude, we observe that since, by Lemma 11.2, for all $n \geq e$ the fibers of the maps τ_n are irreducible of dimension $\dim(X)$, we have (3) \Leftrightarrow (4) by the definition of jet codimension. \square

We obtain the following characterization of local rings of finite embedding dimension.

Theorem 11.5. *Let X be a scheme of finite type. For every $\alpha \in X_\infty$, we have*

$$\text{emb.dim}(\mathcal{O}_{X_\infty, \alpha}) < \infty$$

if and only if α is a stable point and is not contained in $(\text{Sing } X)_\infty$.

Proof. If $\alpha \in (\text{Sing } X)_\infty$ then $\text{emb. dim}(\mathcal{O}_{X_\infty, \alpha}) = \infty$ by Theorem 8.9.

Assume then that $\alpha \notin (\text{Sing } X)_\infty$. This implies that X is reduced and irreducible at the generic point $\xi = \alpha(\eta)$. By Corollary 9.5, we can replace X with its irreducible component containing ξ . Then we have $\text{emb. dim}(\mathcal{O}_{X_\infty, \alpha}) = \text{jet. codim}(\alpha)$ by Theorem 8.9, and $\text{jet. codim}(\alpha) < \infty$ if and only if α is stable, by Lemma 11.4. \square

Remark 11.6. By [Reg09, Theorem 2.9], if X is a variety defined over a perfect field of positive characteristic, then X_∞ has finitely many irreducible components only one of which is not contained in $(\text{Sing } X)_\infty$. An example where X_∞ has more than one component is given by the p -fold Whitney umbrella $X = \{xy^p = z^p\} \subset \mathbb{A}^3$ in characteristic p , see [dF, Example 8.1]. Theorem 11.5 implies that if α is the generic point of an irreducible component of X_∞ that is contained in $(\text{Sing } X)_\infty$, then $\mathcal{O}_{X_\infty, \alpha}$ has infinite embedding dimension, and since this ring is zero dimensional, it follows that it has infinite embedding codimension.

From Theorem 11.5, we recover the following fact about maximal divisorial arcs proved in [dFEI08, Theorem 3.8].

Corollary 11.7. *Let X be a reduced scheme of finite type. Then every maximal divisorial arc $\alpha \in X_\infty$ is a stable point.*

Proof. By Theorem 10.4, the local ring $\mathcal{O}_{X_\infty, \alpha}$ has finite embedding dimension, and hence α is a stable point by Theorem 11.5. \square

One of the nice features of local rings of finite embedding dimension comes from the following elementary observation.

Lemma 11.8. *For any scheme Z over a field, the completion $\widehat{\mathcal{O}_{Z, z}}$ of the local ring of Z at a point z is Noetherian if and only if $\text{emb. dim}(\mathcal{O}_{Z, z}) < \infty$.*

Proof. This is in fact a general result about completions of local rings. Let $(\widehat{R}, \widehat{\mathfrak{m}})$ be the \mathfrak{m} -adic completion of a local ring (R, \mathfrak{m}) . If $\mathfrak{m}/\mathfrak{m}^2$ is finite dimensional, then $\widehat{\mathfrak{m}}$ is finitely generated by [Stacks, Tag 0315], and this implies that \widehat{R} is Noetherian. The converse follows by the fact that since $\widehat{\mathfrak{m}}^2 \subset \widehat{\mathfrak{m}}^2$, there is always a surjection $\widehat{\mathfrak{m}}/\widehat{\mathfrak{m}}^2 \rightarrow \mathfrak{m}/\mathfrak{m}^2$. \square

The following property is an immediate consequence of Theorem 11.5.

Corollary 11.9. *Let X be a reduced scheme of finite type. The completion*

$$\widehat{\mathcal{O}_{X_\infty, \alpha}}$$

of the local ring at a point $\alpha \in X_\infty$ is Noetherian if and only if α is a stable point and is not contained in $(\text{Sing } X)_\infty$.

Proof. From Theorem 11.5 and Lemma 11.8. \square

The fact that the completion of the local ring at a stable point $\alpha \in X_\infty \setminus (\text{Sing } X)_\infty$ is Noetherian is a result of Reguera. It follows from [Reg06, Corollary 4.6], which proves that $\widehat{\mathcal{O}_{(X_\infty)_{\text{red}}, \alpha}}$ is Noetherian (cf. [MR, §2.3, (vii)]), and [Reg09, Theorem 3.13], which proves that there is an isomorphism $\widehat{\mathcal{O}_{X_\infty, \alpha}} \simeq \widehat{\mathcal{O}_{(X_\infty)_{\text{red}}, \alpha}}$. Notice that this last result of Reguera is stated in characteristic 0, but the proof extends to all perfect fields using Hasse–Schmidt derivations.

The Curve Selection Lemma [Reg06, Corollary 4.8] easily follows from Cohen’s Structure Theorem once one knows that these rings are Noetherian. It is a powerful statement that allows the study of certain containments among sets in the arc space via the use of

arcs in the arc space. All the current proofs solving the Nash problem in dimension 2 [FdBPP12, dFD16] use the Curve Selection Lemma in an essential way.

Even though it is not explicitly stated there, the proof of [Reg06, Corollary 4.8] requires that the stable point is not contained in the arc space of the singular locus of the variety. The following example shows that requiring that α is not contained in $(\text{Sing } X)_\infty$ is a necessary condition.

Example 11.10. Let $X = \{xy^2 = z^2\} \subset \mathbb{A}^3$, and consider the 1-jet $\gamma = (t, 0, 0) \in X_1$. A direct computation shows that the set $W = (\pi_0^{-1}(\gamma))_{\text{red}}$ is an irreducible constructible subset of X_∞ that is contained in $(\text{Sing } X)_\infty$. It follows that the generic point α of W is a stable point but the local ring $\mathcal{O}_{X_\infty, \alpha}$ has infinite embedding dimension by Theorem 8.9, and hence the completion of this ring is not Noetherian by Lemma 11.8.

We conclude with the following property which was obtained by different methods in [Reg09, Proposition 4.1]. As before, one should bear in mind that the statement in [Reg09] implicitly assumes that the stable points are not contained in the arc spaces of the singular loci. A more general statement in characteristic 0 which applies to maps that are not necessarily birational is proven in [Reg09, Proposition 4.5].

Corollary 11.11. *Let $f: Y \rightarrow X$ be a proper birational morphism between reduced schemes of finite type over a perfect field. Then the induced map $f_\infty: Y_\infty \rightarrow X_\infty$ induces a bijection*

$$\{\text{stable points } \beta \in Y_\infty \setminus (\text{Sing } Y)_\infty\} \xrightarrow{1-1} \{\text{stable points } \alpha \in X_\infty \setminus (\text{Sing } X)_\infty\}.$$

Proof. By Corollary 9.4 and Theorem 11.5. □

Notice, by contrast, that the image $f_\infty(V)$ of a constructible set $V \subset Y_\infty$ needs not be constructible in X_∞ . This is shown in the next example.

Example 11.12. Let $f: Y \rightarrow X$ be the blow-up of a smooth closed point $x \in X$ of a variety of dimension at least 2, $E \subset Y$ the exceptional divisor, and $y \in E$ a closed point. The set $V \subset Y_\infty$ of arcs with positive order of contact with E at points in $E \setminus \{y\}$ is constructible, but its image $f_\infty(V) \subset X_\infty$ is not constructible, since it is equal to $W \setminus \bigcup_{i \geq 1} Z_i$ where W is the set of arcs through x and Z_i is the set of arcs with order i at x and principal tangent direction equal to y .

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